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INTEGRAL SOLUTIONS OF TERNARY QUADRATIC DIOPHANTINE EQUATION

$$5x^2 + 2y^2 = 7z^2$$

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ABSTRACT:

The Diophantine equation of degree two with three unknowns represented by $5x^2 + 2y^2 = 7z^2$ is investigated for its non-zero distinct integral solutions. A few interesting relations between the values of $x, y\ z$ and special numbers are presented.

KEYWORDS:

Ternary Quadratic equation, Integral Solutions, Factorization Method.

INTRODUCTION:

A ternary quadratic Diophantine equation is an algebraic equation involving three variables, each appearing with degree two, where only integer solutions are considered. Such equations play an important role in number theory because they help in understanding relationships between integers and their structural properties. The equation $5x^2 + 2y^2 = 7z^2$ is a homogeneous quadratic Diophantine equation, and the main objective is to determine whether it admits a non-trivial integer solutions and to study the method used to obtain them.

METHODS OF ANALYSIS:

PROBLEM:

The Ternary Quadratic Diophantine equation to be solved for its non-zero distinct integral solution is

$$5x^2 + 2y^2 = 7z^2 \quad (1)$$

Three different patterns of solutions of (1) are illustrated below:

Pattern 1:

$5x^2 + 2y^2 = 7z^2$

Using linear transformation,

$$x = u + 2v \; ; y = u - 5v \quad (2)$$

To substitute (2) in (1)

$5(u + 2v)^2 + 2(u - 5v)^2 = 7z^2$

$5(u^2 + 4v^2 + 4uv) + 2(u^2 + 25v^2 - 10uv) = 7z^2$

$5u^2 + 20v^2 + 20uv + 2u^2 + 50v^2 - 20uv = 7z^2$

$5u^2 + 2u^2 + 20v^2 + 50v^2 = 7z^2$

$7u^2 + 70v^2 = 7z^2$

Dividing by 7,

$$u^2 + 10v^2 = z^2 \quad (3)$$

Changing above equation into a parametric form,

$u^2 + (\sqrt{10}\, v)^2 = z^2$

By using the solutions of Pythogorean Diophantine equation

$u = 10m^2 - n^2 v = 2mn$

$z = 10m^2 + n^2$, where m and n are integers.

Substituting u, v values in (2),

$x = (10m^2 - n^2) + 2(2mn) = 10m^2 - n^2 + 4mn$

$y = (10m^2 - n^2) - 5(2mn) = 10m^2 - n^2 - 10mn$

$z = 10m^2 + n^2$

Therefore,

$$x = 10m^2 - n^2 + 4mn$$
$$y = 10m^2 - n^2 - 10mn$$
$$z = 10m^2 + n^2$$

Hence integer solution of equation (1) is obtained.

Properties:

1) $u^2 + 10v^2 \equiv u^2 (mod\ 5)$
2) $u^2 + 10v^2 \equiv u^2 (mod\ 2)$
3) $u^2 + 10v^2 \equiv u^2 (mod\ 10)$
4) $u^2 + 10v^2 \equiv u^2 + 2v^2 (mod\ 4)$
5) $u^2 + 10v^2 \equiv u^2 + 3v^2\ (mod\ 7)$

Pattern 2:

From Pattern I, we reduce the equation:

$u^2 = z^2 - 10v^2$

Suppose that

$$u = a^2 - 10b^2 \qquad (4)$$

Using (4) equation in the above equation

$$\left(a + \sqrt{10}b\right)^2 \left(a - \sqrt{10}b\right)^2 = \left(+\sqrt{10}v\right)\left(z - \sqrt{10}v\right) \qquad (5)$$

coefficients on both sides of equation (5), we get

$z = a^2 + 10b^2, v = 2ab$

From equation (2):

$$x = u + 2v\,, y = u - 5v \qquad (6)$$

Substitute $u = a^2 - 10b^2\,, v = 2ab$ in (6)

$x = u + 2v$

$= (a^2 - 10b^2) + 4ab$

$x = a^2 + 4ab - 10b^2$

$y = u - 5v$

$y = (a^2 - 10b^2) - 10ab \quad = a^2 - 10ab - 10b^2$

$x = a^2 + 4ab - 10b^2$

$y = a^2 - 10ab - 10b^2$

$z = a^2 + 10b^2$

Properties:

1) $x = a^2 + 4ab - 10b^2 \equiv a^2 + 4ab \ (mod\ 5)$
2) $z = a^2 + 10b^2 \equiv a^2 (mod\ 5)$
3) $z = a^2 + 10b^2 \equiv a^2 + 3b^2 \ (mod\ 7)$
4) $x^2 + 2y^2 \equiv 0 \ (mod\ 7)$
5) $x^2 + 2y^2 = z^2 \ (mod\ 7)$

Pattern 3:

Write the reduced equation from 1:

$u^2 + 10v^2 = z^2$

$$u^2 + 10v^2 = z^2 * 1 \qquad (7)$$

$u^2 = z^2 - 10v^2$

Assume that

$$z = a^2 + 10b^2 \qquad (8)$$

Write identity 1 in the quadratic ring (z i√10) using the norm:

$3^2 + 10 \cdot 2^2 = 9 + 40 = 49 = 7^2$

$$1 = (3 + 2i\sqrt{10})(3 - 2i\sqrt{10}) / 49 \qquad (9)$$

Using equations (7), (8), (9) and employing the method of factorization:

$$u + i\sqrt{10}\, v = 1/7\,(3 + 2i\sqrt{10})(a + i\sqrt{10}\, b)^2 \qquad (10)$$

Compute:

$(a + i\sqrt{10}\, b)^2 = (a^2 - 10b^2) + i\sqrt{10}\,,(2ab)$

Multiply by the unit $3 + 2i\sqrt{10}/7$ and compare real and $i\sqrt{10}$ parts to obtain

$u = 1/7\,\{3(a^2 - 10b^2)\} - 20(2ab)$

$$u = 1/7\,(3a^2 - 40ab - 30b^2) \qquad (11)$$

$v = 1/7\,\{2(a^2 - 10b^2)\} + 3(2ab)$

$$v = 1/7\,(2a^2 + 6ab - 20b^2) \qquad (12)$$

Our need is integer solutions, so choose a,b.

So the numerator of (11) & (12) are divisible by 7.

$Put\ a = 7A\,, b = 7B\ in$ (11) & (12)

$u = 7(3A^2 - 40AB - 30B^2)$

$= 21A^2 - 280AB - 210B^2$

$v = 7(2A^2 + 6AB - 20B^2)$

$= 14A^2 + 42AB - 140B^2$

$z = 49A^2 + 10B^2$

Comparing integer solution of x, y, z

$x = u + 2v$

$= 49A^2 - 196AB - 490B^2$

$y = u - 5v$

$= -49A^2 - 490AB + 490B^2$

$z = 49A^2 + 10B^2$

Properties :

1 .) $x + y \equiv 0\ (mod\ 2)$

2.) $x - y \equiv 0\ (mod\ 2)$

3.) $x \equiv 49A^2 + 49B^2 \equiv 0\ (\ mod\ 7)$

4.) $x \equiv y \equiv 0\ (mod\ 14)$

5.) $x \equiv y \equiv 0\ (49)$

The above equation represents non-zero distinct integral solution of equation (1) on two parameters.

CONCLUSION:

The equation $5x^2 + 2y^2 = 7z^2$ is a quadratic Diophantine equation, where we seek integer solutions for $x, y,$ and z that satisfy this equation. Such equations are important in number theory and have applications in areas like algebraic number theory, cryptography, and quadratic forms.

REFERENCES:

[1] Cohen, H. (2007). Number Theory, Vol. I: Tools and Diophantine Equations. Springer.

[2] Dickson, L. E. (1952). History of the Theory of Numbers, Vol. II: Diophantine Analysis. Chelsea Publishing Company.

[3] Mordell, L. J. (1969). Diophantine Equations. Academic Press.

[4] Niven, I., Zuckerman, H. S., & Montgomery, H. L. (1991). An Introduction to the Theory of Numbers (5th ed.). Wiley.

[5] Nagell, T. (1981). Introduction to Number Theory. Chelsea Publishing Company.

[6] Ribenboim, P. (2001). Classical Theory of Algebraic Numbers. Springer.

[7] Dr. V. Pandichelvi and Mrs. R. Vanaja, "A Paradigm for Two Classes of Silmultaneous Exponential Diophantine Equations", Indian Journal of Science and Technology, Volume 16, Issue 40, pg:3514-3521.Issue 40, pg : 3514-3521.

[8] Smart, N. P. (1998). The Algorithmic Resolution of Diophantine Equations. Cambridge University Press.

[9] Vanaja, R., & Pandichelvi, V. (2024). Significance of continued fraction to solve binary quadratic equations. Indian Journal of Scientific Engineering and Science, 8(2), 1–6.

[10] Vanaja, R., Rajanila, R., & Pugazharasi, M. (2025). Integral solutions of ternary quintic Diophantine equations $x^2 + y^2 = 17z^2$. International Research Journal of Modernization in Engineering Technology and Science, 7(1), 5447–5449.

[11] Vanaja, R., Pugazharasi, M., & Rajanila, R. (2025). Integral solutions of ternary quintic Diophantine equations $x^2 + y^2 = 10z^2$. International Research Journal of Education and Technology, 7(1), 333–338.

[12] Vanaja, R., & Rajanila, R. (2025). Integral solutions of ternary quintic Diophantine equations $x^2 + y^2 = 37z^2$. International Research Journal of Research Publication and Reviews, 6(2), 4776–4779.